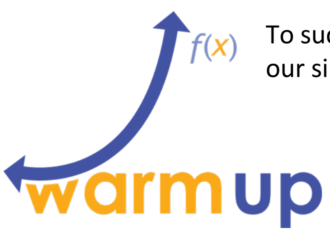
Unit 3 EXPONENTIAL and LOGARITHMIC FUNCTIONS

3.1 Exponential Expressions and Equations

Background Skills – Exponent Rules



To successfully complete this unit, and even enjoy it, we must first brush up our sills on a concept last thoroughly visited in Math 10C – **exponents**.

Exponent

$3^4 = 3 \times 3 \times 3 \times 3$

Base — *Appears 4 times*

Exponent Rules *(Remember these?)*

Name	Rule	Example (simplify each)
Product of Powers	$(b^m)(b^n) = b^{m+n}$	$x^3 \times x^4 =$
Quotient of Powers	$\frac{b^m}{b^n} = b^{m-n}$	$\frac{x^5}{x^3} =$
Power of a Power	$(b^m)^n = b^{mn}$	$(y^3)^2 =$
Power of a Product	$(ab)^m = a^m b^m$	$(4x^4)^2 =$
Power of a Quotient	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{2x}{y^2}\right)^3 =$
Zero Exponent	$b^0 = 1$	$(7y^2 \times 2y^6)^0 =$
Negative Exponents	$b^{-n} = \frac{1}{b^n}$ or $\frac{1}{b^{-n}} = b^n$	$\frac{1}{2^{-4}} =$
Neg. Exp. *fraction base*	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	$\left(\frac{2}{3}\right)^{-2} =$
Rational Exponents	$b^{\frac{m}{n}} = \sqrt[n]{b^m}$ or $(\sqrt[n]{b})^m$	$8^{\frac{2}{3}} =$

Answers are at the bottom of the next page

← Try Each First!

Don't peek!

Evaluate each of these three, try w/o using your calc!

Worked Example Simplify Each of the Following Expressions — RTD Learning PowerMath

(a) $(2x^3y^2)(-3x^4y^3)^2$ **(b)** $\left(\frac{2x^3y^4}{10x^2y^6}\right)^3$ **(c)** $\left(\frac{12m^{-3}n^{-1}}{8m^2n^{-2}}\right)^{-2}$

(a) Start with 2nd bracket, apply exp. "2"

$= (2x^3y^2)(9x^8y^6)$

$(-3)^2$ $(x^4)^2$ $(y^3)^2$

Square everything inside, one at a time

Answer:

$= \mathbf{18x^{11}y^8}$

2×9 $x^3 \times x^8$ $y^2 \times y^6$ *add exponents*

(b) Start by simplifying "inside"

$= \left(\frac{x}{5y^2}\right)^3$

$10 \div 2$

Higher exp. of x is on top (3 vs. 2). So subtract exponents, start w/ top
→ **exp.** of x is: $\mathbf{3-2}$

Higher exp. of y is on bottom to start. So subtract exponents, start w/ bottom
→ **exp. of y is: $6-4$**

Then, apply outside exp., "**3**", to both the numerator and denom.:

$= \frac{(x)^3}{(5y^2)^3}$

$= \frac{\mathbf{x^3}}{\mathbf{125y^6}}$

(c) Start by simplifying "inside"

Exp. of m is *higher on bottom*. So start w/ n term, which is *higher on top*.
→ **exp. of n is: $-1-(-2)$**

$= \left(\frac{3n^1}{2m^5}\right)^{-2}$

12/8 reduces

Exp. of m is higher on bottom.
→ **exp. of m is: $2-(-3)$**

Next, apply the rule: $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{-n}$

$= \left(\frac{2m^5}{3n}\right)^2 = \frac{(2m^5)^2}{(3n)^2}$

$= \frac{\mathbf{4m^{10}}}{\mathbf{9n^2}}$

note: Final answers should not have *negative exponents*

Class Example 3.11 *Simplifying Expressions with Exponents*

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Simply Each of the following:

(a) $(5x^5y^3)^2$ (b) $(-4m^3n^2)^2(10mn^3)$ (c) $\left(\frac{12a^2b^3}{6b^4}\right)^3$ (d) $\left(\frac{x^{-2}y^3}{2x^{-1}y^{-2}}\right)^{-3}$

Class Example 3.12 *Simplifying Expressions with Exponents*

Evaluate Each of these: (Try without a calculator!)

(a) 8^{-2} (b) $\left(\frac{2}{3}\right)^{-3}$ (c) $8^{\frac{5}{3}}$ (d) $9^{-0.5}$ (e) $25^{-\frac{3}{2}}$

Exponent Rules Table – Answers *(in order down the table)*

$\mathbf{x^7}$ (add exp.s) $\mathbf{x^2}$ (subt.) $\mathbf{y^6}$ (mult) $4^2 \times (x^4)^2 = \mathbf{16x^8}$ $\frac{(2x)^3}{(y^2)^3} = \frac{\mathbf{8x^3}}{\mathbf{y^6}}$ $= \mathbf{1}$ (zero exp.) $\frac{1}{\frac{1}{2^4}} = 2^4 = \mathbf{16}$ $\left(\frac{3}{2}\right)^2 = \frac{\mathbf{9}}{\mathbf{4}}$ $(\sqrt[3]{8})^2 = (2)^2 = \mathbf{4}$

Solving Exponential Equations

An **exponential equation** has the variable in the exponent, for example: $2^{3x-4} = 32$

One method to solve involves writing all terms in the same base. Let's see how that works here…

$2^{3x-4} = 2^{\square}$

We need to re-write 32 as a power of 2. (Since the left side is a power of 2)

→ **32 can be written as 2^5**

$2^{3x-4} = 2^5$

Now our equation consists of two power terms of the same base.

$3x - 4 = 5$

Since the expressions are equal, and the bases are equal, then **the exponents must also be equal.**

$3x = 9$

$x = 3$

To solve an **exponential equation**:

❶ Re-write all terms with the *same base*
❷ If necessary, simplify using Exponent Rules, so there is a single term on each side
❸ Set the exponents equal and solve

Worked Example

Simplify the Exponential Equation: $9^{x-7} = 27^{2x-9}$

Algebraic Solution:

Re-write 9 and 27 using the **same base**

$\square^{x-7} = \square^{2x-9}$

$(3)^2 \qquad (3)^3$

So equation becomes:

$(3^2)^{x-7} = (3^3)^{2x-9}$

First simplify each side....

$3^{2x-14} = 3^{6x-27}$ *multiply exponents*

Once both sides are fully simplified, ***set the exponents equal***

$2x - 14 = 6x - 27$

$-4x = -13$

$x = \dfrac{13}{4}$

Verify numerically on your calc:

```
NORMAL FLOAT AUTO REAL RADIAN MP
9^((13/4)-7)
                2.639918926E-4
27^(2(13/4)-9)
                2.639918926E-4
```

Substitute $x = 13/4$ into **both sides** of orig. equation

✓ Same funky exp. notation decimal on both sides!

→ *So our solution is **verified***

Solve graphically on your calculator:

One method is to:

1) Set Equation to zero: $9^{x-7} - 27^{2x-9} = 0$

2) Graph $y_1 =$ *left side*, $y_2 =$ *right side*

3) Sol. is the x-coord. of pt. of intersect

```
Plot1  Plot2  Plot3
\Y1=9^(X-7)-27^(2X-9)
\Y2=0
```

2^{nd} + TRACE to get CALC menu

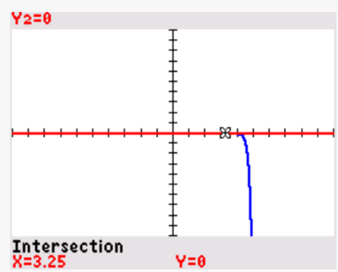
```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```

Window Note:

We don't need a "perfect" window setting – it's not crucial to even see the graphs!

```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
```

So long as the solution is within these values, we're good. (If not, adjust)



Class Example 3.13 *Solving Exponential Equations*

For the equation: $8^{x+2} = \left(\dfrac{1}{4}\right)^{x+3}$

(a) Solve algebraically

(b) Verify numerically and solve graphically on your calc.

Answers from previous page

3.11 (a) $25x^{10}y^6$ (b) $160m^7n^7$ (c) $\dfrac{8a^6}{b^3}$ (d) $\dfrac{8x^3}{y^{15}}$ **3.12** (a) $\dfrac{1}{64}$ (b) $\dfrac{27}{8}$ (c) 32 (d) $\dfrac{1}{3}$ (e) $\dfrac{1}{125}$

Worked Example *Algebraically solve the Exponential Equation:* $\dfrac{8^{x+1}}{4^{2x-1}} = \left(\dfrac{1}{8}\right)^{x-3}$

Algebraic Solution: Re-write 8, 4, and 1/8 using the **same base**

$$\frac{\square^{x+1}}{\square^{2x-1}} = (\square)^{x-3}$$

So equation becomes: $\dfrac{(2^3)^{x+1}}{(2^2)^{2x-1}} = (2^{-3})^{x-3}$

Simplify each side.... $\dfrac{2^{3x+3}}{2^{4x-2}} = 2^{-3x+9}$

*Then on Left Side, **subtract exponents*** $\dfrac{b^m}{b^n} = b^{m-n}$

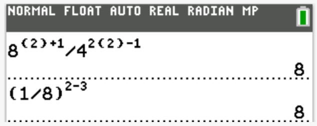
$2^{3x+3-4x+2} = 2^{-3x+9}$

set the exponents equal

$-x + 5 = -3x + 9$

$2x = 4$ ➔ $x = 2$

Verify numerically:



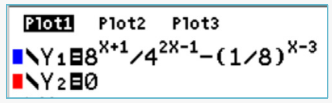
Substitute $x = 2$ into orig. equation

✓ Same value on both sides!

➔ *So our solution is **verified***

CALC TIP

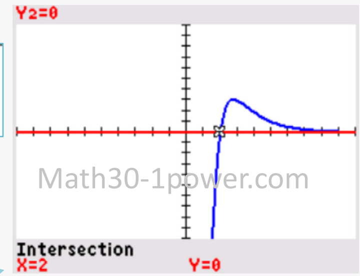
Solve graphically on calc:



Graph: $9^{x-7} - 27^{2x-9} = 0$

y_1 = *left side* y_2 = *right side*

Solution is the x-coord. of the pt. of intersect



Class Example 3.14 *Solving Exponential Equations of three terms*

For the equation: $4^{x-1} = 8^{(2x+2)}16^{(x-3)}$

(a) Solve algebraically

(b) Verify numerically on your calc.

(c) Solve graphically on your calc.

Class Example 3.15 *Solving Equations where the base is unknown*

Exam Style

Algebraically determine the value of x in the exponential equation $\left(\dfrac{a}{b}\right)^{(4x-1)} = \left(\dfrac{b^3}{a^3}\right)^{(x+2)}$, where $a \neq b$, $a \neq 0$, and $b \neq 0$.

Answers from previous page

3.13 (a) $x = -12/5$ (b) ***Verify numerically:*** Substitute $-12/5$ for x in both sides, each works out to approx. 0.435 ➔ $8^{-12/5+2}$

*To **verify graphically**, sketch on calc $y_1 = 8^{x+2}$ and $y_2 = (1/4)^{x+3}$, confirm graphs match!*

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Class Example 3.16 *Solving Equations where the variable is in the base*

Algebraically solve the equation $2(3x-1)^{-\frac{3}{4}} = 16$, and verify your solution.

Worked Example — Solve: $5x^{\frac{3}{2}} = 135$

Sol.:

$\frac{5x^{\frac{3}{2}}}{5} = \frac{135}{5}$ — *First – isolate the power term, $x^{\frac{3}{2}}$*

$x^{\frac{3}{2}} = 27$ — *Goal: Get exp. of "x" to 1.*

Multiplies to 1

$\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = (27)^{\frac{2}{3}}$ — *Take both sides to the exp. 2/3 (reciprocal)*

$x^1 = (27)^{\frac{2}{3}}$

$x = 9$

Applications of Exponential Equations

Finally we explore real world situations that can be modeled using exponential equations – where some initial value (a) has a multiplication factor (b) applied every certain period of time (p).

Worked Example — The number of bacteria in a sample is shown to triple every 7 hours. Initially, there are 9 colonies present. Set up and solve an equation to determine the time it would take for the number of colonies to reach 2187.

Algebraic Solution:

On formula sheet: $y = a(b)^{\frac{t}{p}}$

- t / p: *Period of time for mult. factor to be applied*
- y: *end amount*
- a: *initial amount*
- b: *Multiplication growth (or decay) factor* — *GROWTH: $b > 1$*, *DECAY: $0 < b < 1$*

The mult. factor (b) is 3 *(population triples)*

The end amount (y) is 2187

The initial amount (a) is 9

And p is 7 hrs *(time it takes for population to triple)*

$$2187 = 9(3)^{\frac{t}{7}}$$

$$243 = (3)^{\frac{t}{7}}$$ *Isolate power term*

$$3^5 = (3)^{\frac{t}{7}}$$ *Re-write 243 as a power of 3*

$$5 = \frac{t}{7}$$

$$t = 35 \text{ hrs}$$

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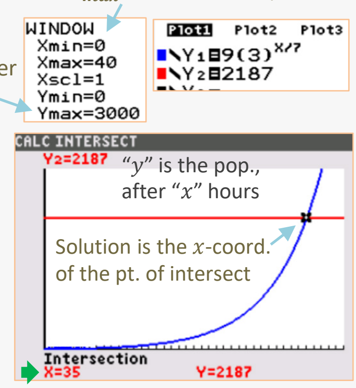
Solve on calc:

Use $x_{max} = 0$ for "time" problems

Use y_{max} greater than 2187



"y" is the pop., after "x" hours

Solution is the x-coord. of the pt. of intersect

Class Example 3.17 *Applications of Exponential Equations*

A particularly strong investment fund has doubled in value over the past 5 years. Assuming that the fund continues this performance, setup and algebraically solve an equation to determine how long it would take for a \$5 000 investment to grow to \$80 000.

Answers from previous page

3.14 (a) $x = 1/2$ (b) ***Verify numerically:*** Substitute $1/2$ for x in both sides, each works out to 0.5 → $4^{1/2-1}$ → $8^{2(1/2)+2} \times 16^{1/2-3}$ (c) *To verify* ***graphically***, *sketch on calc* $y_1 = 4^{x-1}$ *and* $y_2 = 8^{2x+2} \times 16^{x+3}$, *standard window, confirm graphs match!*

3.15 $x = -5/7$ RTD Learning PowerMath

3.1 Practice Questions

1. Fully simply each of the following expressions:

(a) $(5x^2y^3)(3xy^{-1})^2$

(b) $\dfrac{(5x^2y^3)(8x^3y^2)}{-10x^5y^2}$

(c) $\left(\dfrac{4m^2n^4}{-6m^5n^2}\right)^3$

(d) $\left(\dfrac{2x^2y^{-4}}{x^{-1}y^2}\right)^{-2}$

2. Evaluate each, showing simplification steps: *Try first without a calculator, use your calc to verify!*

(a) $\left(-\dfrac{3}{4}\right)^{-2}$

(b) $81^{\frac{3}{4}}$

(c) $\left(\dfrac{25}{16}\right)^{-\frac{3}{2}}$

(d) $\dfrac{2^{-3}}{\left(\dfrac{8}{50}\right)^{-\frac{1}{2}}}$

3. Use an algebraic process to solve each of the following equations. Verify your answers.

(a) $6^{3-3n} = \dfrac{1}{216}$

(b) $64^{2x-3} = 16$

(c) $4^{3x} = 8^{x+1}$

(d) $(9)^{x-4}(3)^{2x-1} = 27^{x+1}$

Answers from previous page

3.16	$x = 17/48$	**4.17**	**20 years**	RTD Learning PowerMath

4. Algebraically solve each of the following equations. Verify by graphing on your calculator.

(a) $5(25)^{2x+1} = (125)^{x-2}\left(\frac{1}{5}\right)^{x-1}$ (b) $\left(\frac{1}{9}\right)^{x-2} = \frac{3^{2x+1}}{27^{x-3}}$ (c) $(3x+2)^{-\frac{2}{3}} = \frac{1}{16}$

(d) $\left(\frac{125}{216}\right)^{-\frac{x}{2}} = \left(\frac{6}{5}\right)^{3x+2}$ (e) $\left(\frac{2}{3}\right)^{2x} = \left(\frac{27}{8}\right)^{x-2}$ (e) $\left(\frac{1}{4}\right)^{x-2} = \frac{16^{2x+1}}{8^{x+1}}$

Answers to Practice Questions on the previous page

1. (a) $45x^4y$ (b) $-4y^3$ (c) $-\frac{8n^6}{27m^9}$ (d) $\frac{y^{12}}{4x^6}$ **2.** (a) $\frac{16}{9}$ (b) 27 (c) $\frac{64}{125}$ (d) $\frac{1}{20}$

3. (a) 2 (b) $11/6$ (c) 1 (d) 12 RTD Learning PowerMath

5. A student used an algebraic process to solve the equation $\frac{3^{x^2+x}}{27^{3x-1}} = 3\left(\frac{1}{9}\right)^{x-2}$. He is able to simplify the equation to $x^2 + bx + c = 0$, where $b, c \in I$

The value of c is:

Exam Style

A. -8

B. -4

C. -2

D. -1

6. The foundation of a house has approximately 1200 termites. The terminate population is doubling every 20 days. Set up and algebraically solve an equation to determine how long it would take for the termite population to reach approximately 153 600.

7. An adult takes 400 mg of Ibuprofen. The half-life for the amount of Ibuprofen in a person's system is 3 hours. *Note: This means that after 3 hrs, half of the original amount ingested remains in the body.*

Set up and algebraically solve an equation to determine how long it would take for the amount of Ibuprofen in the person's body to decrease to 12.5 mg.

Answers to Practice Questions on the previous page ***and this page***

4. (a) -4 (b) -6 (c) $62/3$ (d) $-4/3$ (e) $6/5$ (f) $3/7$ RTD Learning PowerMath

5. C **6.** 140 days **7.** 15 hrs

3.2 Graphs of Exponential Functions

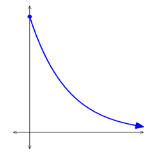
We saw in the previous section how exponential equations involved terms where the variable is in the exponent. **Exponential functions** are can be used to model many real-world situations.

- *The world population*
- *The value of an investment earning positive interest*
- *The measured amount of a decaying radioactive isotope*
- *The value of a used car*
- *The temperature of a cup of hot chocolate as it cools*

Exponential functions can model any of these, given certain parameters.

We'll dive further into applications in section 4.7, after learning a bit about logarithms. For now we'll focus on the basic graphs.

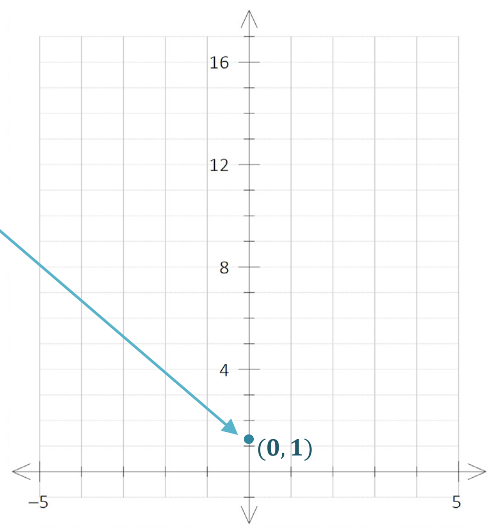
*These three involve **negative exponential growth**, or **exponential decay*** ➔



Exploration #1 The Graph of $y = 2^x$

1 ➡ Complete the table of values below, and plot the remaining points on the graph. Then, sketch the smooth curve that goes through each of your points. Then – you will have graphed your first exponential function.

x	2^x
0	$2^0 = 1$
1	
2	
3	
4	
-1	
-2	
-3	



2 ➡ ***Function Essentials:***

State each of the following

Domain

Range

Asymptote

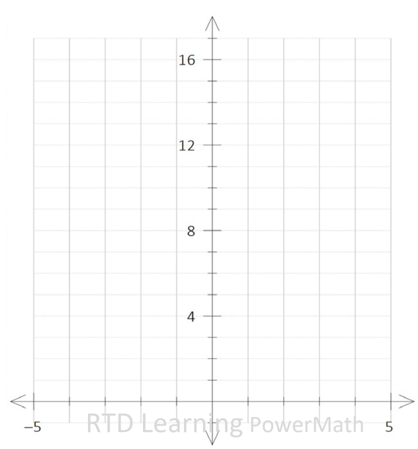
x-intercept

y-intercept

The Graph of $y = (\frac{1}{2})^x$

Next we sketch the graph of the function obtained by **horizontally reflecting** the graph of $y = 2^x$, about the line $x = 0$.

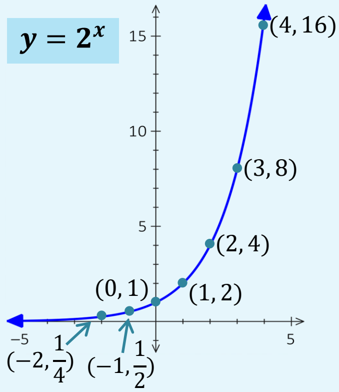
3 ➡ Use transformations to show that the resulting function equation is: $y = \left(\frac{1}{2}\right)^x$



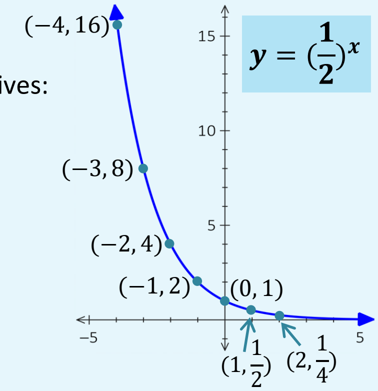
4 ➡ ***Function Essentials:***

Are there any differences from the graph of $y = 2^x$?

We introduce the *basic exponential function* as:



Applying a *horizontal reflection* gives:



For any graph, $y = b^x$ $\quad b > 0, b \neq 1$:

Domain: $\{x \in \mathbb{R}\}$ *y-intercept:* $(0, 1)$ *Asymptote:* **Horizontal Asymptote at $y = 0$**

Range: $\{y > 0, y \in \mathbb{R}\}$ *x-intercept:* $(1, 0)$ ↑ *As x gets larger and larger the graph approaches, but never touches, the x-axis.*

Note how the characteristics; domain, range, intercepts and asymptote – are the same for any base of $y = b^x$.

And about that base, ***b****....*

- ☞ It can't be "0" *Or else y would just be 0 for every x .... $y = 0^x$*
- ☞ It can't be "1" *Or else y would just be 1 for every x .... $y = 1^x$*
- ☞ It can't be negative *Note that f we allowed negative bases, any even value of x would return a positive value*

So we define the base to be: $b > 0, b \neq 1$:

Now with that, let's explore the effect of changing the base:

Exploration #2 Comparing Graphs in the form $y = b^x$, $b > 1$, for different b values

Analyze the graphs on the right. All points with integer coordinates are shown.

1 ➡ For each equation listed to the right, indicate the number of the matching graph.

______ $y = \left(\frac{3}{2}\right)^x$

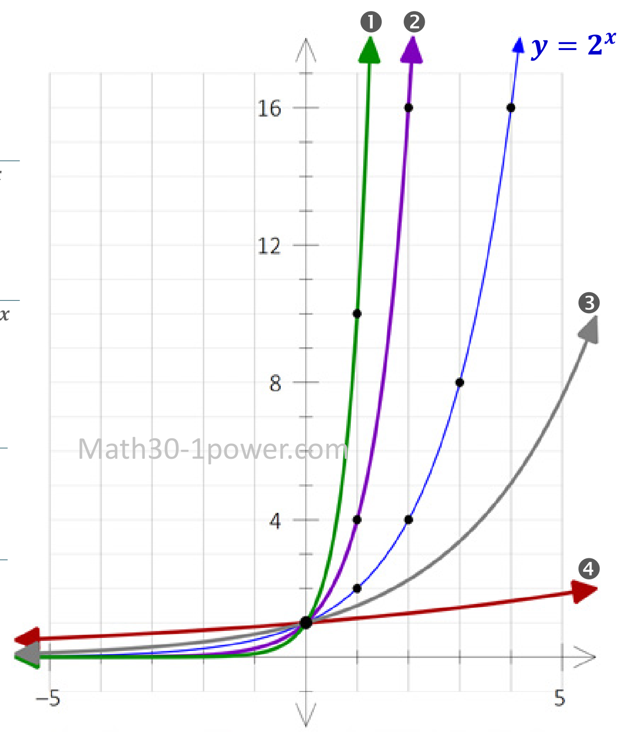
______ $y = \left(\frac{10}{9}\right)^x$

______ $y = 4^x$

______ $y = 10^x$

2 ➡ Describe the effect on the graph of $y = b^x, b > 1$, as b gets larger.

3 ➡ Describe the effect on the graph of $y = b^x, b > 1$, as b gets *closer to 1*.



Exploration #3 Comparing Graphs in the form $y = b^x$, where b is between 0 and 1

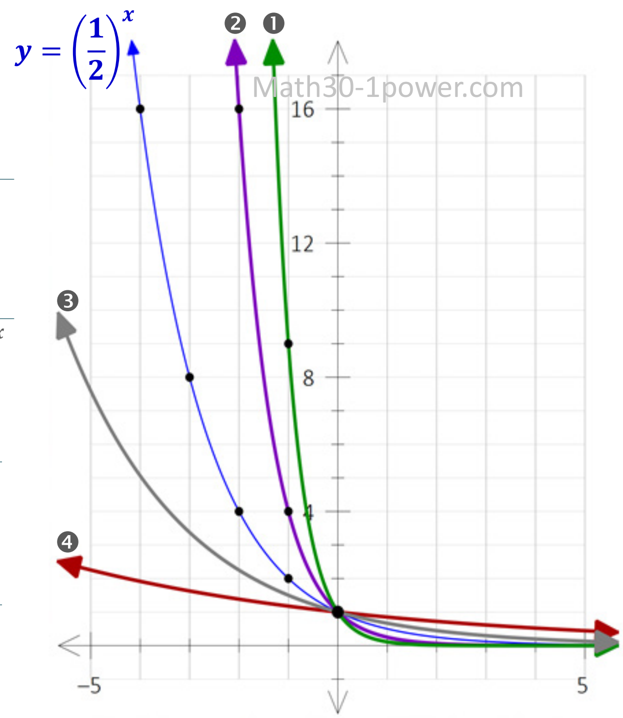
Analyze the graphs on the right. All points with integer coordinates are shown.

1 ➡ For each equation on the right, indicate the number of the graph that matches.

2 ➡ Describe the effect on the graph of $y = b^x$, $0 < b < 1$, as b gets *closer to 0.*

3 ➡ Describe the effect on the graph of $y = b^x$, $0 < b < 1$, as b gets *closer to 1.*

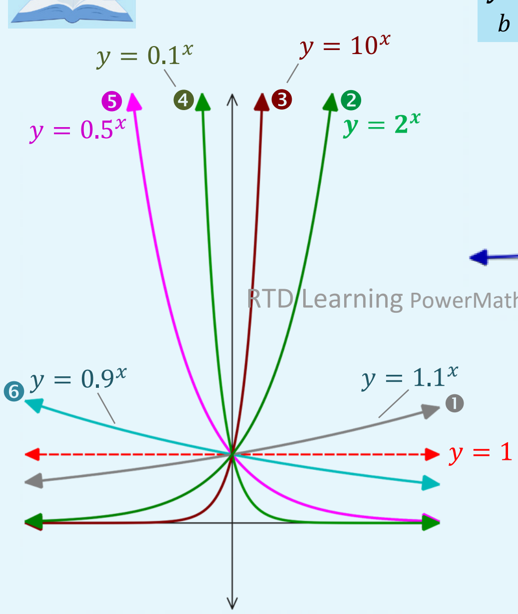
$y = \left(\frac{2}{3}\right)^x$
$y = \left(\frac{9}{10}\right)^x$
$y = \left(\frac{1}{4}\right)^x$
$y = \left(\frac{1}{10}\right)^x$



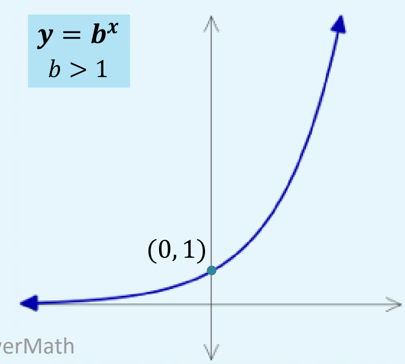
Note that for all of these graphs::

$\{x \in \mathbb{R}\}$	$\{y > 0, y \in \mathbb{R}\}$	H.A. at $y = 0$	$y = 1$	n/a
Domain	Range	Asymptote	y-intercept	x-intercept

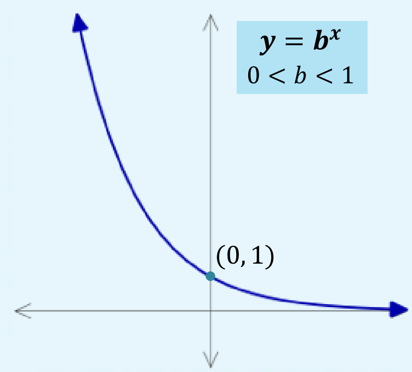
Given the graph of $y = b^x$ ...



If $b > 1$, the graph ***increases***



If $0 < b < 1$, the graph ***decreases***



If $b > 1$, the graph *bends-up* from the horizontal line $y = 1$

The larger the base (the further from "1"), the greater the increase... compare graphs ❶,❷, and ❸ on the left.

The smaller the base (the further from "1"), the greater the decrease... compare graphs ❹, ❺, and ❻ on the left.

We'll next explore the effect of adding a vertical stretch, a, and vertical translation, k, to the graph of $y = b^x$....

Exploration #4

Comparing Graphs of $\boldsymbol{y = a(b)^x}$, $a \neq 0, b > 1$ for different a values

Analyze the graphs on the right. All points with integer coordinates are shown.
The corresponding functions for each of graphs ❶, ❷, and ❸ can be written in the form $y = a(2)^x$.

Hint: The value of "a" can be solved for, or obtained through reasoning, or through exploration with your graphing calculator.

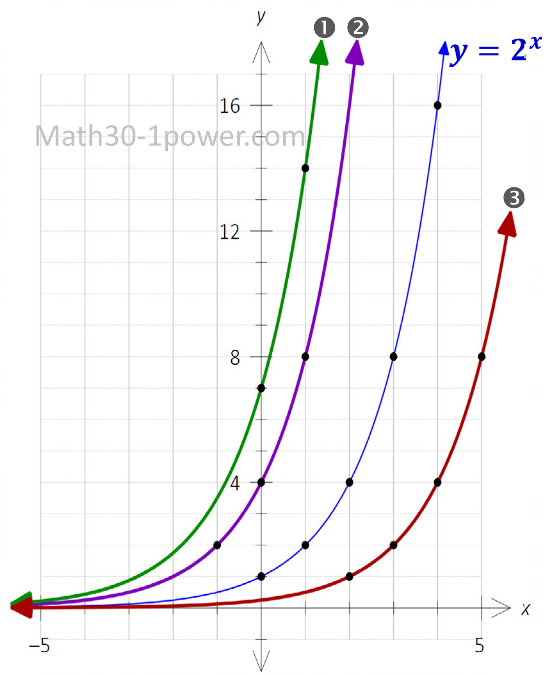
1 ➡ **State the equation** of the function for:

Graph ❶

Graph ❷

Graph ❸

2 ➡ **Describe** the effect of $\boldsymbol{a}$ on the graph of $y = a(b)^x$.



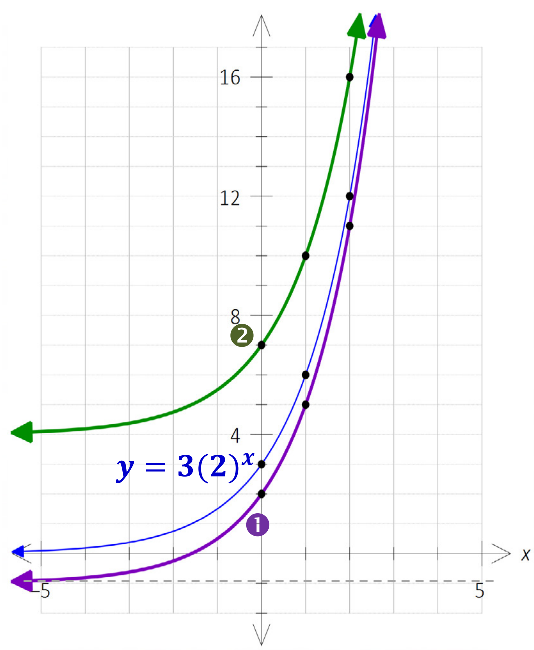
Exploration #5

Effect of Parameter "$\boldsymbol{k}$" in $\boldsymbol{y = a(b)^x + k}$, $a \neq 0, b \neq 1$

Graphs ❶ and ❷ on the right are obtained by applying a vertical translation to the graph of $y = 3(2)^x$.

The horizontal asymptote (HA) for graph ❶ is shown.

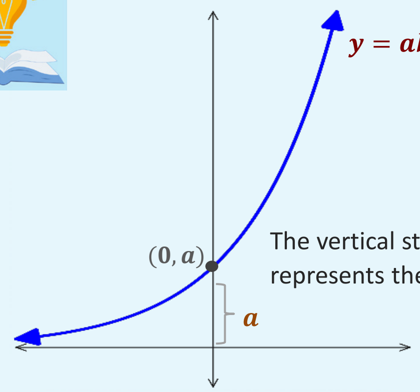
1 ➡ **Explain** how the value $\boldsymbol{k}$, where $a > 0$, affects whether the graph has an x-intercept.



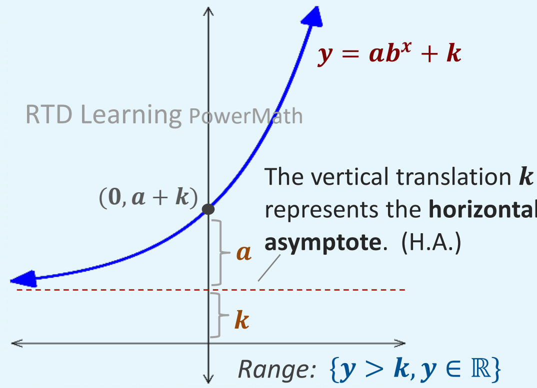
2 ➡ **State the indicated characteristics** for each graph:

	Range	Asymptote	y-intercept
Graph ❶			
Graph ❷			

Given the graph of $y = ab^x$; $a > 0$ ...



Given the graph of $y = ab^x$; $a > 0$ ...



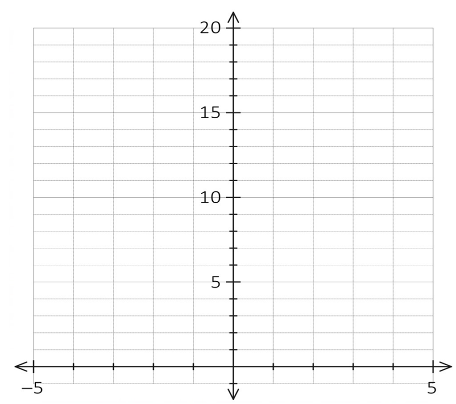
Class Example 3.21 *Determining Graph Characteristics from an Equation*

Given the function $y = 3(2)^x + 4$,

(a) Without graphing, state the range, asymptote, and y-intercept.

(b) Use reasoning to state whether the graph will have an x-intercept. Explain.

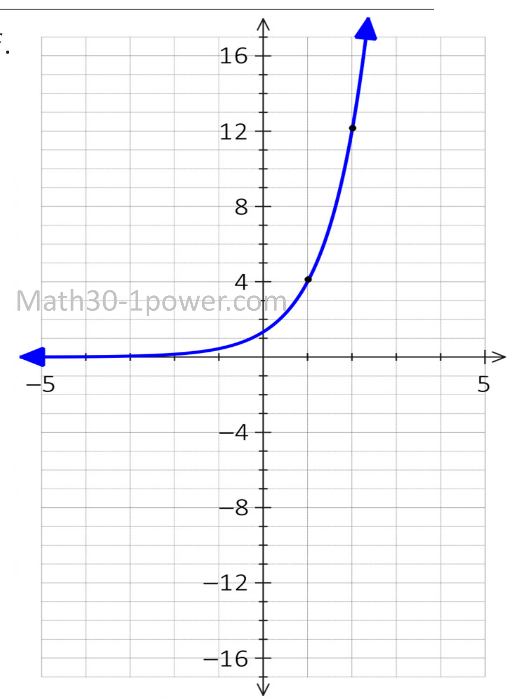
(c) Sketch and label all characteristics.



Exploration #6 The graph of $y = a(b)^x$, $a < 0, b \neq 1$

The graph on the right can each be written in the form $y = a(b)^x$.

1 ➡ Given that the graph has a y-intercept of $\frac{4}{3}$, state the value of a, solve for b, and state the equation.



2 ➡ On the same grid, sketch the graph of the function obtained by vertical reflection about the line $y = 0$. Label it graph ❷.

3 ➡ State the equation of the function that corresponds to each graph.

Exploration #7 Further Exploration of $y = a(b)^x + k$, $a \neq 0, b \neq 1$

The graphs on the right can each be written in the form $y = a(b)^x + k$. All four graphs have the same a and b values.

The horizontal asymptotes are:

$y = 0$ for graph ❶
$y = 4$ for graph ❷
$y = -1$ for graph ❸
$y = -3$ for graph ❹

1 ➡ **Use** graph ❶ to determine the a and b values.

2 ➡ **State an equation** of the function for:

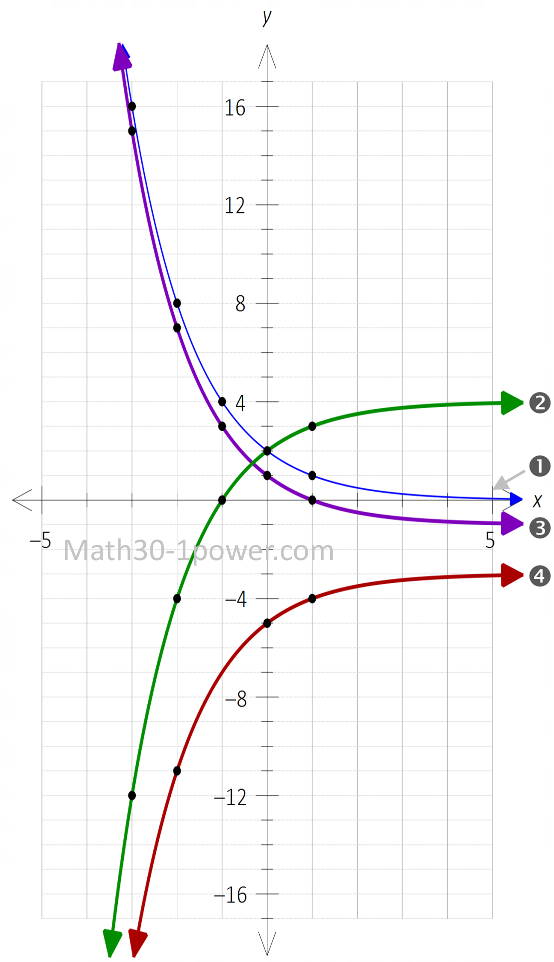
Graph ❶

Graph ❷

Graph ❸

Graph ❹

3 ➡ Use an algebraic process to determine the x-intercept of graph ❷.
(From the equation)



4 ➡ State the range of each graph.

Graph ❶ *Graph* ❷ *Graph* ❸ *Graph* ❹

5 ➡ Given an exponential function in the form $y = a(b)^x + k$, state two possible expressions for the range.

Answers from previous page

3.21 (a) Range is defined by the H.A., which is given by the vertical translation. (4 units up)

Range: $\{y > 4, y \in \mathbb{R}\}$

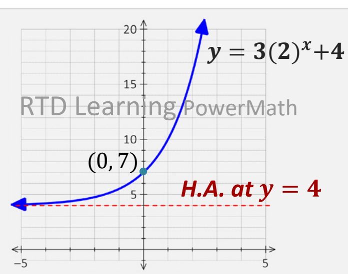
H.A. at $y = 4$

y**-intercept:** $(0, 7)$

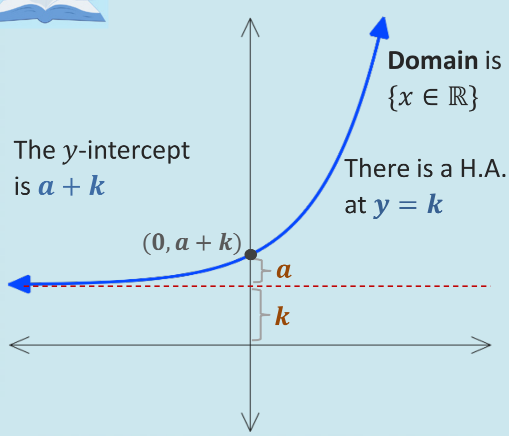
Graph "opens up", as $a > 0$, and the horiz. asymptote (H.A.) is above the x-axis.

➔ **Graph has no x-intercept.**

(c)



An **exponential function** can be written $\boldsymbol{y = a(b)^x + k}$.



- For the y-intercept, set $x = 0$ and evaluate.

  $y = a(b)^{\mathbf{0}} + k$

  $= a(1) + k$ *since* $b^0 = 1$ *for any* b

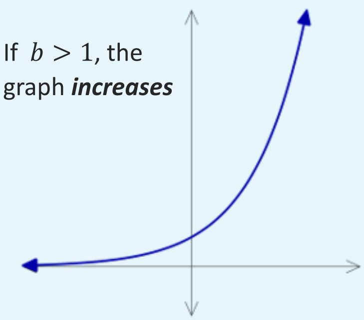
  $= \boldsymbol{a + k}$

- For the x-intercept, set $y = 0$ and solve.

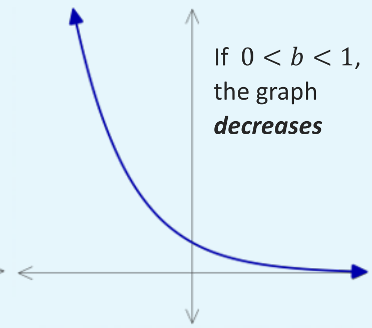
  $\mathbf{0} = a(b)^x + k$

  Solve the resulting equation for the x*-intercept, if it exits.*

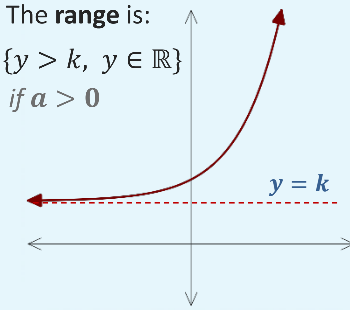
If $b > 1$, the graph ***increases***



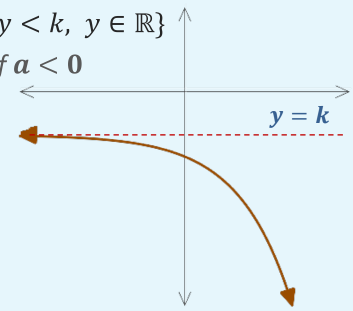
If $0 < b < 1$, the graph ***decreases***



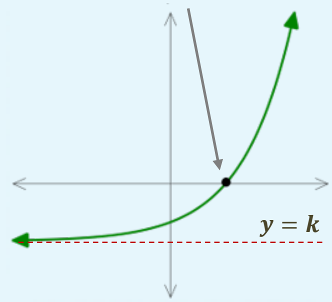
The **range** is: $\{y > k,\ y \in \mathbb{R}\}$ *if* $\boldsymbol{a > 0}$



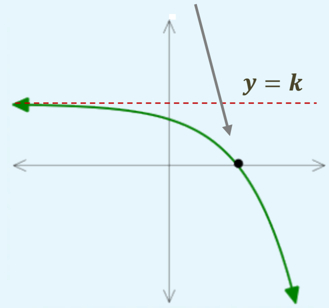
$\{y < k,\ y \in \mathbb{R}\}$ *if* $\boldsymbol{a < 0}$



The graph will have an $\boldsymbol{x}$ **– intercept** if $a > 0$ and $k < 0$



... or if $a < 0$ and $k > 0$



Worked Example

Given the function $y = -4\left(\frac{1}{2}\right)^x + 16$, state the domain, range, asymptote, and any intercepts. Sketch and label all characteristics.

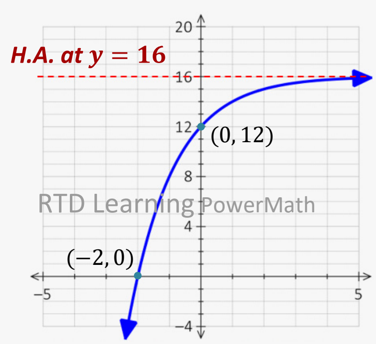
Solution: For all exponential functions (unless restricted by some application), the **DOMAIN** is $\{\boldsymbol{x} \in \mathbb{R}\}$

For **range consider two things:**

☝ The **H.A. is at** $\boldsymbol{y = 16}$ (the vertical translation). So the graph is either entirely above or entirely below $\boldsymbol{y = 16}$.

✌ Since $a < 0$ (there is a negative in front of the function), the graph opens down, and is **entirely below** the line $y = 16$.

So the **RANGE** is $\{\boldsymbol{y < 16},\ \boldsymbol{y} \in \mathbb{R}\}$



$y = -4\left(\frac{1}{2}\right)^x + 16$

Horiz. Asymptote

Base is less than 1, so graph "falls right".
But $a = -4$ *is negative, so* ***vertically reflect***

For $\boldsymbol{y}$**-intercept**, substitute $x = 0$ and evaluate:

$y = -4\left(\frac{1}{2}\right)^{\mathbf{0}} + 16$

$= -4(1) + 16 \Rightarrow = 12$

$\Rightarrow$ $(\mathbf{0}, \mathbf{12})$

For $\boldsymbol{x}$**-intercept**, substitute $y = 0$ and solve:

$\mathbf{0} = -4(\tfrac{1}{2})^x + 16$

$4(\tfrac{1}{2})^x = 16$

$(\tfrac{1}{2})^x = 4$

$(2^{-1})^x = 2^2$

$2^{-x} = 2^2$

$-x = 2$

$x = -2$

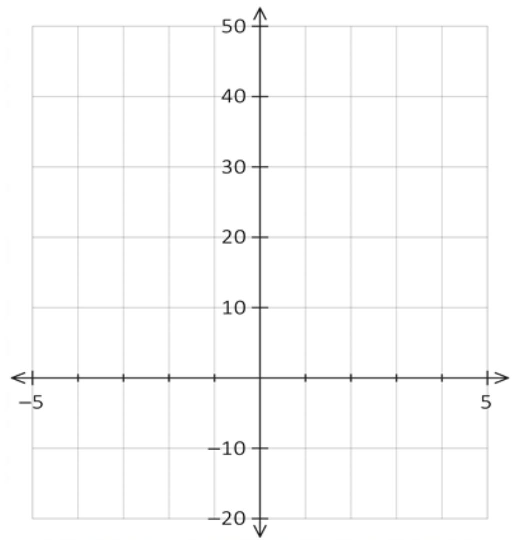
$(\mathbf{-2}, \mathbf{0})$

Class Example 3.22 *Solving Exponential Equations of three terms*

Given the function $y = -3(4)^x + 48$,

(a) State the domain, and analyze the characteristics of equation to determine the range.

(b) Use an algebraic process to determine the x and y intercepts. *Verify using your graphing calculator.*



Class Example 3.23 *Identifying graphs in the form $y = ab^x$*

Each graph on the right represents an exponential function that can be written in the form $y = a(b)^x$; $b > 1$

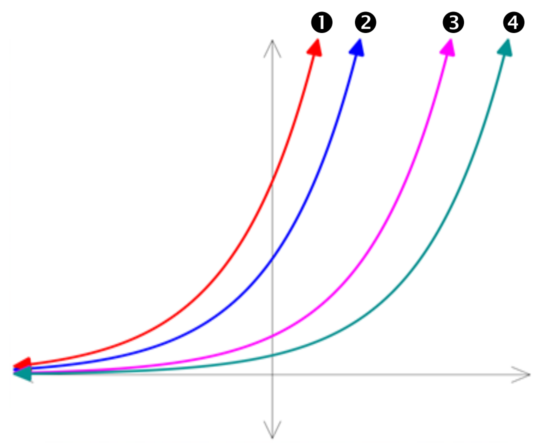
Use reasoning to match each equation with a graph number.

(a) $y = b^x$ ________

(b) $y = \frac{2}{5}(b)^x$ ________

(c) $y = 5b^x$ ________

(d) $y = 4b^x$ ________



Class Example 3.24 *Identifying graphs in the form $y = b^x$*

Each graph on the right represents an exponential function that can be written in the form $y = (b)^x$; $b \neq 1$

Use reasoning to match each equation with a graph number.

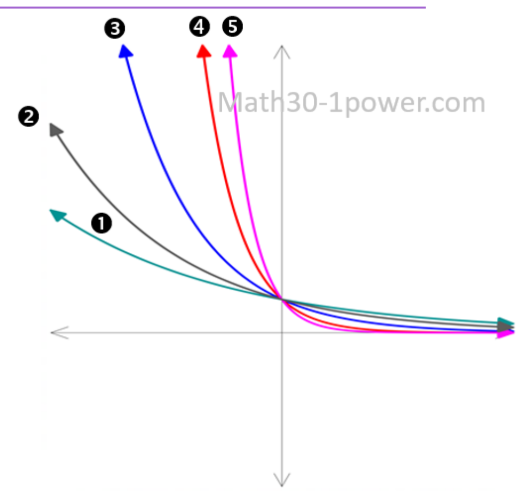
(a) $y = \left(\frac{1}{2}\right)^x$ ________

(b) $y = \left(\frac{1}{8}\right)^x$ ________

(c) $y = \left(\frac{1}{4}\right)^x$ ________

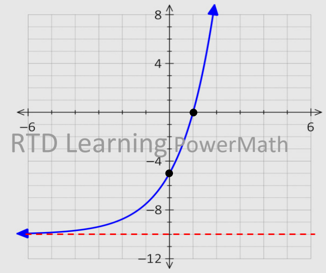
(d) $y = \left(\frac{2}{3}\right)^x$ ________

(e) $y = \left(\frac{3}{4}\right)^x$ ________



Worked Example

On the graph shown the horizontal asymptote and points (•) shown have integer values. Determine an equation for the corresponding are exponential function, in the form $y = a(b)^x + k$.



Solution:

Start by identifying the H.A. (horizontal asymptote), which defines the $\boldsymbol{k}$ value. ➔ $\boldsymbol{k = -10}$

So we have $\boldsymbol{y = a(b)^x - 10}$

Next, the value of $\boldsymbol{a}$ (vertical stretch) can be counted as the distance from the ***H.A.*** *and the* ***y-intercept****.* ➔ $\boldsymbol{a = 5}$

So now we have $\boldsymbol{y = 5(b)^x - 10}$

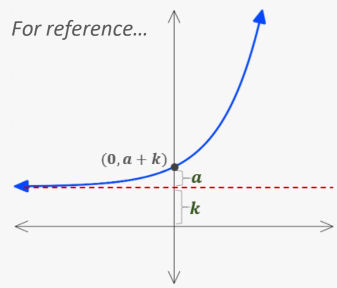
Finally, use any other point on the graph, such as $(1, 0)$, to solve for $\boldsymbol{b}$*.*

$$\boldsymbol{0} = 5(b)^{\boldsymbol{1}} - 10$$
$$10 = 5(b)$$
$$b = 2$$

So now that we have k, a, and b, we can state the equation:

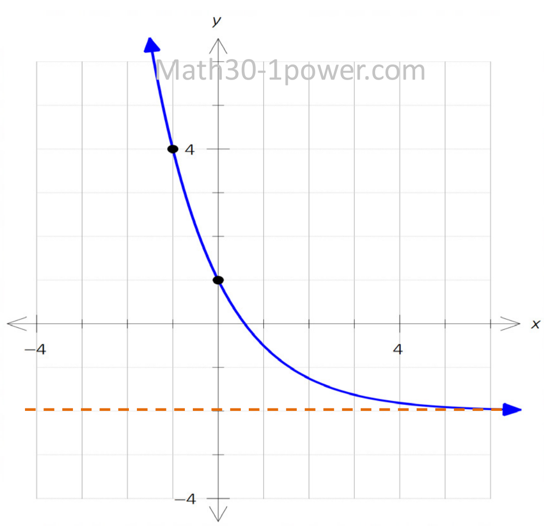
$$\boxed{\boldsymbol{y = 5(2)^x - 10}}$$

For reference...



Class Example 3.25 *Solving Exponential Equations of three terms*

For each graph below, the horizontal asymptote and points indicated (•) are all integer values. Determine an equation for the corresponding are exponential function, in the form $y = a(b)^x + k$.



Answers from previous page

3.22 (a) Domain is $\{x \in \mathbb{R}\}$ ← Graph is entirely below the H.A. (since $\boldsymbol{a} = \boldsymbol{-3}$ is negative)

Range is $\{y < \boldsymbol{48}, y \in \mathbb{R}\}$

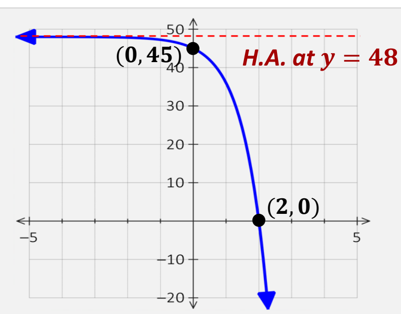
H.A. at $\boldsymbol{y = 48}$ ← H.A. is given by the vertical translation

(b) x int: $(\boldsymbol{2}, \boldsymbol{0})$ ← For x-int, set $y = 0$ and solve

$\boldsymbol{0} = -3(4)^x + 48$
$3(4)^x = 48$
$4^x = 16$
$4^x = 4^{\boldsymbol{2}}$ ➔ $\boldsymbol{x = 2}$

y int: $(\boldsymbol{0}, \boldsymbol{45})$ ← $y = -3(4)^{\boldsymbol{0}} + 48$
$= -3 + 48$

(c)



3.23 (a) ❸ (b) ❹ (c) ❶ (d) ❷ **4.24** (a) ❸ (b) ❺ (c) ❹ (d) ❷ (e) ❶